Numerical Aspects of the Method of Moments in Anisotropic Materials

Ignace Bogaert [#1]

[#] *Department of Information Technology, Ghent University*
Sint-Pietersnieuwstraat 41, 9000 Ghent, Belgium
[1] Ignace.Bogaert@intec.UGent.be

***Abstract*—Currently, boundary integral equations are predominantly used for the solution of scattering problems involving isotropic components. However, in this contribution, the extension of boundary integral equations to anisotropic components is considered. The main problems arising from this are the computation of the Green's dyadics and the evaluation of the impedance integrals. Recent progress towards solving these issues is presented.**

I. Introduction

Boundary integral equations (BIEs) are widely used for the rigorous modeling of (frequency domain) electromagnetic scattering problems. When they are applicable, i.e. when the scatterer consists of a number of homogeneous regions, BIEs constitute a powerful alternative to finite element or finite difference methods. This is due to the fact that BIEs only introduce unknown current densities on the boundaries of the homogeneous regions, while the other methods typically introduce unknown fields in the volume. This difference also results in quicker convergence of the error for BIEs (as a function of mesh density) and easier handling of singular solutions (e.g. at corners).

However, despite these advantages, BIEs are only seldomly used for scatterers consisting of homogeneous regions filled with *anisotropic* materials. Such materials occur for example in circulators and other microwave components. In cases where integral equations are used for anisotropic materials, it usually concerns either a specific material for which closed-form expressions for the Green's dyadics exist [1] or it concerns the use of a volume integral equation [2], [3]. The reason for this is the considerable additional complexity that is introduced in the Method of Moments (MoM) [4], when it is applied to general anisotropic materials. The main sources of complexity are:

- the lack of closed-form expressions for the Green's dyadics in general anisotropic materials, which necessitates the numerical evaluation of the Green's dyadics,
- the lack of accurate ways to compute the impedance integrals, especially the singular ones.

When considering fast solvers, some further difficulties are the extension of the current fast matrix-vector multiplication methods (such as the MLFMA [5], [6]) and the extension of effective preconditioners [7] to the anisotropic case. These complications are quite serious on their own and, when combined, form a severe impediment to the use of BIEs for anisotropic materials. In contrast to BIEs, finite element methods are easily extended to anisotropic materials [8] because they do not rely on Green's dyadics.

This contribution focusses on the two complications associated with the pure MoM, i.e. the computation of the Green's dyadics and the impedance integrals. Firstly, explicit expressions of the Green's dyadics in terms of derivatives of the bianisotropic scalar Green's function (BSGF) [9], [10] are presented. Secondly, based on these expressions, a frequency-independent matrix is constructed such that the components of the Green's dyadics can be computed with a simple and elegant matrix multiplication. Lastly, these expressions are used to shed light on the structure of the singularities occurring in the Green's dyadics. This is important for the future development of numerical computation strategies for the impedance integrals.

II. The Green's Dyadics

In this Section, the Green's dyadics in anisotropic materials will be briefly introduced. Much of this material can also be found in [11], [9], [10], but it is useful to fix the notation and to supply some background information.

Maxwell's curl equations in the frequency domain and in a general linear anisotropic material are given by

$$\boldsymbol{\nabla} \times \boldsymbol{h}(\boldsymbol{r}) - j\omega\bar{\bar{\varepsilon}} \cdot \boldsymbol{e}(\boldsymbol{r}) = \boldsymbol{j}\delta(\boldsymbol{r}), \tag{1a}$$

$$-\boldsymbol{\nabla} \times \boldsymbol{e}(\boldsymbol{r}) - j\omega\bar{\bar{\mu}} \cdot \boldsymbol{h}(\boldsymbol{r}) = \boldsymbol{m}\delta(\boldsymbol{r}). \tag{1b}$$

Vectors are denoted in bold, 3 by 3 dyadics are denoted with a double bar and $\boldsymbol{r}$ is the Cartesian position vector. If elementary dipole sources

$$\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{j}\delta(\boldsymbol{r}), \tag{2a}$$

$$\boldsymbol{m}(\boldsymbol{r}) = \boldsymbol{m}\delta(\boldsymbol{r}), \tag{2b}$$

are taken as the right hand side in Maxwell's equations, the solution is given in terms of the Green's dyadics

$$\boldsymbol{e}(\boldsymbol{r}) = \bar{\bar{G}}_{ee}(\boldsymbol{r}) \cdot \boldsymbol{j} + \bar{\bar{G}}_{em}(\boldsymbol{r}) \cdot \boldsymbol{m}, \tag{3a}$$

$$\boldsymbol{h}(\boldsymbol{r}) = \bar{\bar{G}}_{me}(\boldsymbol{r}) \cdot \boldsymbol{j} + \bar{\bar{G}}_{mm}(\boldsymbol{r}) \cdot \boldsymbol{m}. \tag{3b}$$

Using spatial Fourier transforms, it can be shown [9], [10] that

$$\begin{bmatrix} \bar{\bar{G}}_{ee}(\boldsymbol{r}) & \bar{\bar{G}}_{em}(\boldsymbol{r}) \\ \bar{\bar{G}}_{me}(\boldsymbol{r}) & \bar{\bar{G}}_{mm}(\boldsymbol{r}) \end{bmatrix} = j\omega^2 \mathsf{A}\left(-j/\omega\boldsymbol{\nabla}\right) G(\omega\boldsymbol{r}), \quad (4)$$

with the frequency-independent BSGF

$$G(\boldsymbol{r}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \frac{e^{j\boldsymbol{s}\cdot\boldsymbol{r}}}{D(\boldsymbol{s})} \mathrm{d}\boldsymbol{s}, \quad (5)$$

and

$$\mathsf{A}(\boldsymbol{s}) = \mathrm{Adj}\left[\mathsf{P}(\boldsymbol{s})\right], \quad (6)$$

$$D(\boldsymbol{s}) = \mathrm{Det}\left[\mathsf{P}(\boldsymbol{s})\right], \quad (7)$$

$$\mathsf{P}(\boldsymbol{s}) = \begin{bmatrix} \bar{\bar{\varepsilon}} & -\boldsymbol{s}\times\mathbb{1} \\ \boldsymbol{s}\times\mathbb{1} & \bar{\bar{\mu}} \end{bmatrix}, \quad (8)$$

$$\boldsymbol{s}\times\mathbb{1} = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}. \quad (9)$$

In equation (6), the $\mathrm{Adj}\left[\cdot\right]$ operator computes the adjugate matrix of its argument, which can be defined as

$$\mathrm{Adj}\left[\mathsf{P}(\boldsymbol{s})\right] = \mathsf{P}^{-1}(\boldsymbol{s})\mathrm{Det}\left[\mathsf{P}(\boldsymbol{s})\right]. \quad (10)$$

The components of the matrix $\mathsf{A}(\boldsymbol{s})$ are polynomials of degree at most four in the components of $\boldsymbol{s}$, which why the substitution

$$\boldsymbol{s} \to -\frac{j}{\omega}\boldsymbol{\nabla}, \quad (11)$$

in equation (4) can be performed without problems.

The problem of computing the BSGF has previously been considered in the literature. For example, [12] gives a number of Green's functions for various types of fourth-order partial differential equations. Also, a fully numerical approach for computing and storing the BSGF and its derivatives has been proposed in [9], [10]. This approach is applicable to all lossy bianisotropic materials [13]. Therefore, the computation of the BSGF will be considered solved in this contribution, such that the attention can be turned towards the analysis of $\mathsf{A}(\boldsymbol{s})$.

III. Explicit Expressions for the Green's Dyadics

To compute the 6 by 6 matrix $\mathsf{A}(\boldsymbol{s})$, it is useful to split it into four 3 by 3 subblocks, given by

$$\mathsf{A}(\boldsymbol{s}) = \begin{bmatrix} \mathsf{A}_{11}(\boldsymbol{s}) & \mathsf{A}_{12}(\boldsymbol{s}) \\ \mathsf{A}_{21}(\boldsymbol{s}) & \mathsf{A}_{22}(\boldsymbol{s}) \end{bmatrix}. \quad (12)$$

Replacing $\mathsf{A}(\boldsymbol{s})$ with such a block in equation (4) yields an expression for the corresponding Green's dyadic, e.g.

$$\bar{\bar{G}}_{ee}(\boldsymbol{r}) = j\omega^2 \mathsf{A}_{11}\left(-j/\omega\boldsymbol{\nabla}\right) G(\omega\boldsymbol{r}). \quad (13)$$

Each block is calculated by means of a Schur complement. The inverses occurring in these Schur complements are then evaluated using an operation known from dyadic analysis: the double cross product. It can be defined in terms of the adjugate as follows

$$\bar{\bar{a}}^{\times}_{\times}\bar{\bar{b}} = \frac{1}{2}\left[\mathrm{Adj}\left[\bar{\bar{a}}+\bar{\bar{b}}\right] - \mathrm{Adj}\left[\bar{\bar{a}}\right] - \mathrm{Adj}\left[\bar{\bar{b}}\right]\right]^T. \quad (14)$$

Using this operation, omitting a lengthy calculation, the four blocks can be expressed as

$$\begin{aligned} \mathsf{A}_{11}(\boldsymbol{s}) = {} & \boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\mu}} \cdot \boldsymbol{s}\boldsymbol{s}^T \\ & + 4\left[\bar{\bar{\mu}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \cdot \bar{\bar{\mu}} \cdot \left[\bar{\bar{\varepsilon}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \\ & - 2\boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\mu}} \cdot \left[\bar{\bar{\mu}}^{\times}_{\times}\bar{\bar{\varepsilon}}\right]^T + \mathrm{Det}\left[\bar{\bar{\mu}}\right]\mathrm{Adj}\left[\bar{\bar{\varepsilon}}\right], \quad (15) \\ \mathsf{A}_{22}(\boldsymbol{s}) = {} & \boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\varepsilon}} \cdot \boldsymbol{s}\boldsymbol{s}^T \\ & + 4\left[\bar{\bar{\varepsilon}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \cdot \bar{\bar{\varepsilon}} \cdot \left[\bar{\bar{\mu}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \\ & - 2\boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\varepsilon}} \cdot \left[\bar{\bar{\varepsilon}}^{\times}_{\times}\bar{\bar{\mu}}\right]^T + \mathrm{Det}\left[\bar{\bar{\varepsilon}}\right]\mathrm{Adj}\left[\bar{\bar{\mu}}\right], \quad (16) \\ \mathsf{A}_{12}(\boldsymbol{s}) = {} & \left[\bar{\bar{\varepsilon}}^{\times}_{\times}\bar{\bar{\varepsilon}}\right]^T \cdot \left[\boldsymbol{s}\times\mathbb{1}\right] \cdot \left[\bar{\bar{\mu}}^{\times}_{\times}\bar{\bar{\mu}}\right]^T \\ & - 2[\boldsymbol{s}\times\mathbb{1}]^{\times}_{\times}\left[\bar{\bar{\varepsilon}}^T \cdot \boldsymbol{s}\cdot\boldsymbol{s}^T \cdot \bar{\bar{\mu}}^T\right], \quad (17) \\ \mathsf{A}_{21}(\boldsymbol{s}) = {} & -\left[\bar{\bar{\mu}}^{\times}_{\times}\bar{\bar{\mu}}\right]^T \cdot \left[\boldsymbol{s}\times\mathbb{1}\right] \cdot \left[\bar{\bar{\varepsilon}}^{\times}_{\times}\bar{\bar{\varepsilon}}\right]^T \\ & + 2[\boldsymbol{s}\times\mathbb{1}]^{\times}_{\times}\left[\bar{\bar{\mu}}^T \cdot \boldsymbol{s}\cdot\boldsymbol{s}^T \cdot \bar{\bar{\varepsilon}}^T\right]. \quad (18) \end{aligned}$$

The terms of degree four have been found in the past (see [14]) but, to the best of the author's knowledge, the lower degree terms have not.

A. Computing the Green's Dyadics

When substituting expressions (15), (16), (17) and (18) into (4), it becomes clear that the components of the Green's dyadics are linear combinations of partial derivatives of the BSGF. Assuming that we have at our disposal a vector $\boldsymbol{w}$ containing these partial derivatives, evaluated in the point $\boldsymbol{v} = \omega\boldsymbol{r}$:

$$\boldsymbol{w} = \begin{bmatrix} G(\boldsymbol{v}) \\ \frac{\partial}{\partial v_x} G(\boldsymbol{v}) \\ \frac{\partial}{\partial v_y} G(\boldsymbol{v}) \\ \frac{\partial}{\partial v_z} G(\boldsymbol{v}) \\ \frac{\partial^2}{\partial v_x^2} G(\boldsymbol{v}) \\ \vdots \\ \frac{\partial^4}{\partial v_z^4} G(\boldsymbol{v}) \end{bmatrix}. \quad (19)$$

Then it becomes possible to compute a matrix M such that

$$\boldsymbol{u} = j\omega^2 \mathsf{M} \cdot \boldsymbol{w}. \quad (20)$$

with $\boldsymbol{u}$ containing the components of the Green's dyadics:

$$\boldsymbol{u} = \begin{bmatrix} \left[\bar{\bar{G}}_{ee}(\boldsymbol{r})\right]_{xx} \\ \left[\bar{\bar{G}}_{ee}(\boldsymbol{r})\right]_{yx} \\ \left[\bar{\bar{G}}_{ee}(\boldsymbol{r})\right]_{zx} \\ \vdots \\ \left[\bar{\bar{G}}_{mm}(\boldsymbol{r})\right]_{zz} \end{bmatrix}. \quad (21)$$

The elements of M can be determined from equations (15) through (18) by identification. For example, the element $\left[\mathsf{M}\right]_{11}$ is given by

$$\left[\mathsf{M}\right]_{11} = \mathrm{Det}\left[\bar{\bar{\mu}}\right]\left\{\left[\bar{\bar{\varepsilon}}\right]_{yy}\left[\bar{\bar{\varepsilon}}\right]_{zz} - \left[\bar{\bar{\varepsilon}}\right]_{zy}\left[\bar{\bar{\varepsilon}}\right]_{yz}\right\}. \quad (22)$$

It is easily seen that M is frequency-independent. In fact, M depends *only* on the material parameters. Therefore, it needs to be computed only once for each material, after which it can be used to easily and elegantly compute the Green's dyadics.

B. Singularities of the Green's Dyadics

For the evaluation of MoM impedance integrals in an anisotropic material, it is crucial to have a solid idea about the structure of the singularity that is present in the Green's dyadics. From the isotropic case, it is known that the Green's dyadics can exhibit a hypersingular, strongly singular and a weakly singular (integrable) part. In the isotropic case, these terms correspond to the fourth order, third order and second and lower order derivatives of the BSGF, respectively. Therefore, to analyze the singularity of the Green's dyadics, the matrices (15), (16), (17) and (18) will be split according to the degree in $\boldsymbol{s}$. For $\mathsf{A}_{11}(\boldsymbol{s})$, this splitting yields

$$\mathsf{A}_{11}(\boldsymbol{s}) \begin{cases} \text{degree } 4: & \boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\mu}} \cdot \boldsymbol{s}\boldsymbol{s}^T \\ \text{degree } 3: & 0 \\ \text{degree} \leq 2: & 4\left[\bar{\bar{\mu}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \cdot \bar{\bar{\mu}} \cdot \left[\bar{\bar{\varepsilon}}^{\times}_{\times}(\boldsymbol{s}\times\mathbb{1})\right]^T \\ & -2\boldsymbol{s}\boldsymbol{s}^T \cdot \bar{\bar{\mu}} \cdot \left[\bar{\bar{\mu}}^{\times}_{\times}\bar{\bar{\varepsilon}}\right]^T + \text{Det}\,[\bar{\bar{\mu}}]\,\text{Adj}\,[\bar{\bar{\varepsilon}}] \end{cases} \tag{23}$$

Apparently, the matrix $\mathsf{A}_{11}(\boldsymbol{s})$ does not have a third degree part. This means that the electric-electric Green's dyadic $\bar{\bar{G}}_{ee}(\boldsymbol{r})$ does not have a strongly singular part, even in the anisotropic case. It does have a hypersingular part, which is generated by the fourth degree term. However, it can be seen that this term is of the form

$$\boldsymbol{s}\underbrace{\left[\boldsymbol{s}^T \cdot \bar{\bar{\mu}} \cdot \boldsymbol{s}\right]}_{\text{scalar}}\boldsymbol{s}^T, \tag{24}$$

such that, when a div-conforming discretization is used, partial integration can be performed to remove two of the derivatives. Therefore, one is left with a weakly singular integral. The same holds trivially for the second degree part of $\mathsf{A}_{11}(\boldsymbol{s})$. Therefore, it can be concluded that a div-conforming discretization of the electric field integral equation in an anisotropic material requires only the evaluation of weakly singular integrals. For $\mathsf{A}_{22}(\boldsymbol{s})$, similar results hold.

For $\mathsf{A}_{12}(\boldsymbol{s})$, the splitting yields

$$\mathsf{A}_{12}(\boldsymbol{s}) \begin{cases} \text{degree } 4: & 0 \\ \text{degree } 3: & -2[\boldsymbol{s}\times\mathbb{1}]^{\times}_{\times}\left[\bar{\bar{\varepsilon}}^T \cdot \boldsymbol{s} \cdot \boldsymbol{s}^T \cdot \bar{\bar{\mu}}^T\right] \\ \text{degree} \leq 2: & \left[\bar{\bar{\varepsilon}}^{\times}_{\times}\bar{\bar{\varepsilon}}\right]^T \cdot [\boldsymbol{s}\times\mathbb{1}] \cdot \left[\bar{\bar{\mu}}^{\times}_{\times}\bar{\bar{\mu}}\right]^T \end{cases} \tag{25}$$

Therefore, no hypersingular integrals occurs and no partial integration needs to be done. However, the third degree part is nonzero such that a strongly singular part exists. From the isotropic case, it is known that this part cannot be made weakly singular by means of partial integration. Instead, it is split into a local, Dirac delta-like part and a weakly singular remainder. It is not immediately clear whether this also holds for the anisotropic case. This is the subject of future research.

IV. Conclusion

Explicit formulas were presented that express the Green's dyadics in terms of the partial derivatives of the bianisotropic Green's function. A matrix formalism was introduced that allows for an easy numerical computation of the Green's dyadic components. Finally, the singularities of the various Green's dyadics were analyzed in the context of the evaluation of impedance integrals. It turns out that the impedance integrals for the electric field integral equation are all convertible to weakly singular integrals. This is not the case for the magnetic field integral equation, as is expected from the isotropic case. Future research is directed towards treating this strongly singular part in a rigorous and numerically satisfactory way.

Acknowledgment

Ignace Bogaert is supported by a postdoctoral grant from the Fund for Scientific Research Flanders (FWO-Vlaanderen).

References

[1] G. Mumcu, K. Sertel, and J. Volakis, "Surface integral equation solutions for modeling 3-d uniaxial media using closed-form dyadic green's functions," *Antennas and Propagation, IEEE Transactions on*, vol. 56, no. 8, pp. 2381 –2388, aug. 2008.

[2] R. Graglia and P. Uslenghi, "Electromagnetic scattering from anisotropic materials, part i: General theory," *Antennas and Propagation, IEEE Transactions on*, vol. 32, no. 8, pp. 867 – 869, aug 1984.

[3] L. Hu, L.-W. Li, and T. Yeo, "Analysis of scattering by large inhomogeneous bi-anisotropic objects using aim," *Progress In Electromagnetics Research*, vol. 99, pp. 21–36, 2009.

[4] R. Harrington, *Field Computations by Moment Methods*. New York: Macmillan, 1968.

[5] W.C. Chew, J. Jin, E. Michielssen, and J. Song, *Fast and Efficient Algorithms in Computational Electromagnetics*. Artech House, 2001.

[6] I. Bogaert, J. Peeters, and F. Olyslager, "A nondirective plane wave MLFMA stable at low frequencies," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 12, pp. 3752–3767, December 2008.

[7] F. Andriulli, K. Cools, H. Bagci, F. Olyslager, A. Buffa, S. Christiansen, and E. Michielssen, "A multiplicative calderon preconditioner for the electric field integral equation," *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 8, pp. 2398–2412, Aug 2008.

[8] L. Valor and J. Zapata, "An efficient finite element formulation to analyze waveguides with lossy inhomogeneous bi-anisotropic materials," *Microwave Theory and Techniques, IEEE Transactions on*, vol. 44, no. 2, pp. 291 –296, feb 1996.

[9] I. Bogaert, "Accurate Computation and Tabulation of the Scalar Green Function for Bi-anisotropic Media and its Derivatives," in *Proceedings of the conference of the Applied Computational Electromagnetics Society*, Columbus, Ohio, USA, April 2012.

[10] ——, "Computing Green's Functions for Bianisotropic Materials," in *Proceedings of the International Conference on Electromagnetics in Advanced Applications*, Cape Town, South Africa, 2–7 September 2012.

[11] B. Jakoby and F. Olyslager, "Asymptotic expansions for greens dyadics in bianisotropic media," *Progress on Electromagnetics Research*, vol. 12, pp. 277–302, 1996.

[12] I. Lindell and F. Olyslager, "Polynomial operators and green functions," *Journal of Electromagnetic Waves and Applications*, vol. 14, no. 8, pp. 1141–1142, 2000.

[13] I. V. Lindell and F. M. Dahl, "Conditions for the parameter dyadics of lossy bianisotropic media," *Microwave and Optical Technology Letters*, vol. 29, no. 3, pp. 175–178, May 2001.

[14] W. Weiglhofer, "Depolarization dyadics," in *Proceedings of the 8th International Conference on Electromagnetics of Complex Media (Bianisotropics 2000), 27-29 September 2000, Lisbon, Portugal*, 2000, pp. 415–420.